## LECTURE 3: INTEGRAL EQUATIONS AND ITS APPROXIMATION

## 1. A BRIEF DISCUSSION ON INTEGRAL EQUATIONS

The theory and application of integral equations (IEs) is an important subject within applied mathematics. IEs are used as mathematical models for many and varied physical situations and occur as reformulations of other mathematical problems. We begin with a brief classification of IEs, then we give some of the classical theory for one of the most popular types of IEs.

### 1.1 Types of integral equations

In classifying IEs, we say roughly that those integral equations in which the integration domain varies with the independent variable in the equation are Volterra IEs (VIEs) and those in which the integration domain is fixed are Fredholm IEs (FIEs).

General form of VIEs of the second kind is

$$
\begin{equation*}
x(t)+\int_{a}^{t} K(t, s, x(s)) d s=y(t), t \geq a \tag{3.1}
\end{equation*}
$$

The function $K(t, s, x(s))$ and $y(t)$ are given, and $x(t)$ is sought. This is a nonlinear IE (NIE), and such equations can be thought of a generalization of

$$
x^{\prime}(t)=f(t, x(t)), t \geq a, x(a)=x_{0},
$$

the initial value problem for ordinary differential equations (ODEs). This equation is equivalent to the IEs

$$
x(t)=x_{0}+\int_{a}^{t} f(s, x(s)) d s, t \geq a
$$

which is a special case of (3.1).

### 1.2 Volterra IEs of the first kind

The general nonlinear VIEs of the first kind has the form

$$
\int_{a}^{t} K(t, s, x(s)) d s=y(t), t \geq a
$$

here $x(s)$ is the unknown function. The general linear VIEs of the first kind is

$$
\begin{equation*}
\int_{a}^{t} K(t, s) x(s) d s=y(t), t \geq a \tag{3.2}
\end{equation*}
$$

For VIEs of the first kind the linear equation is the more commonly studied case. The difficulty of these equations (linear or nonlinear) is that they are "ill-conditioned" to some extent and that makes their numerical solution more difficult.

### 1.3 Abel IEs of the first kind

An important case of (3.2) is the Abel IE

$$
\begin{equation*}
\int_{a}^{t} \frac{H(t, s) x(s)}{\left(t^{p}-s^{p}\right)^{\alpha}} d s=y(t), t \geq a \tag{3.3}
\end{equation*}
$$

here $0<\alpha<1$ and $p>0$ and particularly important case are $p=\{1,2\}$ and $\alpha=1 / 2$. The function $H(t, s)$ is assumed to be smooth. Special numerical methods have been developed for these equations as they occur in a wide variety of applications.

### 1.4 Fredholm IEs of the first and second kind

The general form of second kind of FIE is

$$
\begin{equation*}
\lambda x(t)+\int_{D} K(t, s) x(s) d s=y(t), t \in D, \lambda \neq 0, \tag{3.4}
\end{equation*}
$$

where $D$ a closed bounded set in $R^{m}, m \geq 1$. The kernel function $K(t, s)$ is assumed to be absolutely integrable, and it is assumed to satisfy other properties that are sufficient to imply the Fredholm Alternative Theorem. For $y \neq 0$, we have $\lambda$ and $y$ given, and seek $x$; this is the non-homogeneous problem. For $y=0$ equation (3.4) becomes an eigenvalue problem and seek both the eigenvalue $\lambda$ and the eigen-function $x$.

Fredholm IEs of the first kind has the form

$$
\begin{equation*}
\int_{D} K(t, s) x(s) d s=y(t), t \in D, \lambda \neq 0 \tag{3.5}
\end{equation*}
$$

Example of this type equation occur quite frequently in the subject of potential theory and well study example is

$$
\int_{\Gamma} \log |t-s| x(s) d s=y(t), t \in \Gamma,
$$

with $\Gamma$ a curve in $R^{2}$.

### 1.5 Cauchy singular integral equations (SIEs)

Let $\Gamma$ be an open or closed contour in the complex plane. The general form of a Cauchy SIEs is given by

$$
\begin{equation*}
\lambda a(t) \phi(t)+\frac{b(t)}{\pi i} \int_{\Gamma} \frac{\phi(s)}{s-t} d s+\int_{\Gamma} K(t, s) \phi(s) d s=\psi(t), t \in \Gamma, \tag{3.6}
\end{equation*}
$$

where $a, b, \psi$ and $K$ are given complex-valued functions, and $\phi$ is the unknown function. The function $K$ is to be absolutely integrable and in addition it is to be such that the associated integral operator is a Fredholm integral operator in the sense of Cauchy principal value

$$
\int_{\Gamma} \frac{\phi(s)}{s-t} d s=\lim _{\varepsilon \rightarrow 0}\left(\int_{\Gamma \Gamma_{\varepsilon}} \frac{\phi(s)}{s-t} d s+\int_{\Gamma_{\varepsilon}} \frac{\phi(s)}{s-t} d s\right),
$$

where $\Gamma_{\varepsilon}=\{s \in \Gamma:|s-z| \leq \varepsilon\}$. Cauchy SIEs occur in a variety of physical problems, especially in connection with the solution of PDEs in $R^{2}$.

## 2. DEGENERATE KERNEL METHODS

Integral equations with a degenerate kernel function will be considered in this section for solving general FIEs of the second kind and it is one of the easiest numerical methods to define and analyze.

### 2.1 Compact operators

When $X$ is a finite dimensional vector space and operator $A: X \rightarrow X$ is linear, the equation

$$
A x=y,
$$

has a well-developed solvability theory (this topic mostly related to Linear Algebra).
To extend these results to infinite dimensional space, we introduce the concept of a compact operator $K^{*}$ and then the following section we give theory for operator equations

$$
\begin{equation*}
\left(I-K^{*}\right) x=y \tag{3.7}
\end{equation*}
$$

Definition 3.1: Let $X$ and $Y$ be normed vector spaces, and let $K^{*}: X \rightarrow X$ be linear. Then $K^{*}$ is compact if the set

$$
\left\{K^{*} x:\|x\|_{X} \leq 1\right\}
$$

has compact closure in $Y$. This is equivalent to saying that for every bounded sequence $\left\{x_{n}\right\} \subset X$, the sequence $\left\{K * x_{n}\right\}$ has a subsequence that is convergent to some point in $Y$ , compact operators are also called completely continuous operators.

In the definition, the spaces $X$ and $Y$ need not be complete, but in all applications they are complete therefore we will always assume that $X$ and $Y$ are complete (The completeness of the real numbers, which implies that there are no "holes" in the real numbers, in mathematical analysis a metric space $M$ is called complete (or a Cauchy space) if every Cauchy sequence of points in $M$ has a limit that is also in $M$ or, alternatively, if every Cauchy sequence in $M$ converges in $M$ ).

Let $D$ be a closed bounded set in $R^{m}$, some $m \geq 1$, and define

$$
\begin{equation*}
K * x=\int_{D} K(t, s) x(s) d s, \quad t \in D, x \in C(D) . \tag{3.8}
\end{equation*}
$$

We want to show that $K^{*}: C(D) \rightarrow C(D)$ is both bounded and compact. We assume that $K(t, s)$ is Riemann-integrable as a function of s , for all $t \in D$, and further we assume that the kernel $K(t, s)$ in Eq. (3.8) satisfies the followings

K1: $\lim _{h \rightarrow 0} \omega(h)=0$ with

$$
\omega(h)=\max _{t, \tau \in D} \max _{|t-\tau| \leq h} \int_{D}|K(t, s)-K(\tau, s)| d s .
$$

K2: For the kernel $K(t, s)$ in (3.8) we have

$$
\left\|K^{*}\right\|_{\infty}=\max _{t \in D} \int_{D}|K(t, s)| d s<\infty .
$$

Now we show that operator $K^{*}$ satisfies both assumptions. If $x(s)$ is bounded and integrable, then $K^{*} x(t)$ is continuous with

$$
\begin{align*}
|K * x(t)-K * x(\tau)| & \leq \int_{D}|K(t, s)-K(\tau, s)||x(s)| d s \\
& \leq \max _{t, \tau \in D} \max _{\mid t-\tau \leq \zeta h} \int_{D}|K(t, s)-K(\tau, s)| d s \max _{s \in D}|x(s)| .  \tag{3.9}\\
& =\omega(|t-\tau|)\|x\|_{\infty} .
\end{align*}
$$

Using K2, we have boundedness of $K *$ with

$$
\|K *\|_{\infty}=\max _{t \in D} \int_{D}|K(t, s)| d s<\infty .
$$

Thus we have shown that operator $K^{*}$ is bounded and satisfies both conditions K1 and K2. To show the compactness of $K^{*}$ we first need to identify the compact set in $C(D)$. To do this end, we use Arzela-Ascoli theorem from advanced calculus.

Theorem 3.2 (Arzela-Ascoli theorem): A subset $S \subset C(D)$ has compact closure iff

1) $S$ is a uniformly bounded set of functions
(In mathematics, a bounded function is a function for which there exists a lower bound and an upper bound, in other words, a constant that is larger than the absolute value of any value of this function. If we consider a family of bounded functions, this constant can vary across functions in the family. If it is possible to find one constant that bounds all functions, this family of functions is uniformly bounded).
2) $S$ is an equi-continuous family (In mathematical analysis, a family of functions is equi-continuous if all the functions are continuous and they have equal variation over a given neighbourhood, in a precise sense described herein.
A sequence of functions $f_{n}$ in $C(X)$ is uniformly convergent if and only if it is equi-continuous and converges point-wise to a function. In particular, the limit of an equi-continuous point-wise convergent sequence of continuous functions $f_{n}$ on either metric space or locally compact space is continuous).

Now consider the set $S=\left\{K^{*} x: x \in C(D),\|x\|_{\infty} \leq 1\right\}$. This is uniformly bounded, since

$$
\|K x\|_{\infty} \leq\|K\|\|x\|_{\infty} \leq\|K\| .
$$

In addition, $S$ is equi-continuous from (3.9). Thus $S$ has compact closure in $C(D)$, and operator $K^{*}$ is a compact operator on $C(D)$ to $C(D)$.

In addition let $D=[a, b]$ and consider

$$
\begin{equation*}
K * x(t)=\int_{a}^{b} \log |t-s| x(s) d s \tag{3.10}
\end{equation*}
$$

and

$$
\begin{equation*}
K * x(t)=\int_{a}^{b} \frac{1}{|t-s|^{\beta}} x(s) d s, 0<\beta<1 . \tag{3.11}
\end{equation*}
$$

To show those operators in (3.10) and (3.11) to satisfy conditions K1 and K2 need to rewrite the kernel $K(t, s)$ in the form

$$
\begin{equation*}
K(t, s)=\sum_{i=0}^{p} H_{i}(t, s) L_{i}(t, s) \tag{3.12}
\end{equation*}
$$

for some $p>0$, with each $L_{i}(t, s)$ continuous for $a \leq t, s \leq b$ and each $H_{i}(t, s)$ satisfies K1-K2. For example to show that kernel $K(t, s)$ in (3.10) satisfies K1-K2 need to write

$$
\begin{equation*}
K(t, s)=\log |t-s|=|t-s|^{-1 / 2}\left[|t-s|^{1 / 2} \log |t-s|\right]=H_{1}(t, s) L_{1}(t, s) . \tag{3.13}
\end{equation*}
$$

From (3.13) we can easily see that $H_{1}(t, s)=|t-s|^{-1 / 2}$ satisfies K1-K2 and $L_{1}(t, s)$ is continuous. Thus operator $K^{*}$ is a compact operator on $C[a, b]$ to $C[a, b]$.

Lemma 3.3 (Atkinson [1]): Let $X$ and $Y$ be normed spaces, with $Y$ complete. Let $K^{*} \in L[X, Y]$, and $\left\{K_{n}^{*}\right\}$ be a sequence of compact operators in $L[X, Y]$, and assume $K_{n}^{*} \rightarrow K^{*}$ in $L[X, Y]$, i.e. $\left\|K_{n}^{*}-K^{*}\right\| \rightarrow 0$. Then operator $K^{*}$ is compact.

Lemma 3.4 (Atkinson [1]): Let $K^{*} \in L[X, Y]$ and $L^{*} \in L[X, Y]$, and let $K^{*}$ or $L^{*}$ (or both) be compact. Then $L^{*} K^{*}$ is compact on $X$ to $Z$.

## Integral operator on $L^{2}(a, b)$

Let $X=Y=L^{2}(a, b)$ and let $K^{*}$ be the integral operator defined by (3.8), we first show that under suitable assumptions on kernel $K(t, s)$ the operator $K^{*}: L^{2}(a, b) \rightarrow L^{2}(a, b)$. Let

$$
M=\left[\int_{a}^{b} \int_{a}^{b}|K(t, s)|^{2} d s d t\right]^{1 / 2}<\infty .
$$

For $x \in L^{2}(a, b)$, use the Cauchy-Schwarz inequality to obtain

$$
\begin{align*}
\|K * x\|_{2}^{2} & =\left.\int_{a}^{b} \int_{a}^{b} K(t, s) x(s) d s\right|^{2} d t \\
& \leq \int_{a}^{b}\left[\int_{a}^{b}|K(t, s)|^{2} d s\right]\left[\int_{a}^{b}|x(s)|^{2} d s\right] d t .  \tag{3.14}\\
& =M^{2}\|x\|_{2}^{2}
\end{align*}
$$

This proves that $K x \in L^{2}(a, b)$ and it shows

$$
\|K *\| \leq M .
$$

To examine the compactness of operator $K^{*}$ for more general kernel functions, we assume there is a sequence of the kernels $K_{n}(t, s)$ for which $K_{n}^{*}: L^{2}(a, b) \rightarrow L^{2}(a, b)$ is compact and

$$
\begin{equation*}
M_{n}=\left[\int_{a}^{b} \int_{a}^{b}\left|K(t, s)-K_{n}(t, s)\right|^{2} d s d t\right]^{1 / 2} \xrightarrow{n \rightarrow \infty} 0 \tag{3.15}
\end{equation*}
$$

Now it can be easily shown Cauchy-Schwarz inequality that

$$
\begin{aligned}
\left\|\left(K^{*}-K_{n}^{*}\right) x\right\|^{2} & =\left.\int_{a}^{b} \int_{a}^{b}\left[K(t, s)-K_{n}(t, s)\right] x(s) d s\right|^{2} d t \\
& \left.\leq \int_{a}^{b} \int_{a}^{b}\left|K(t, s)-K_{n}(t, s)\right|^{2} d s\right]\left[\int_{a}^{b}|x(s)|^{2} d s\right] d t=M_{n}^{2}\|x\|_{2}^{2} .
\end{aligned}
$$

This means

$$
\left\|K^{*}-K_{n}^{*}\right\| \leq M_{n} \xrightarrow[n \rightarrow \infty]{ } 0
$$

From Lemma 3.3 this shows operator $K^{*}$ is compact.

### 2.2 General Theory

Consider the IEs of the form

$$
\begin{equation*}
\lambda x(t)-\int_{D} K(t, s) x(s) d s=y(t), t \in D, \lambda \neq 0, \tag{3.16}
\end{equation*}
$$

where $\lambda \neq 0$ and we assume that $D$ is a closed bounded set in $R^{m}, m \geq 1$. We usually work in the space $X=C(D)$ with $\|\cdot\|_{\infty}$ and occasionally in $X=L^{2}(D)$.

In section 2.1, it is shown that the integral operator $K^{*}$ in (3.16) is compact operator on $X$ into $X$. The kernel function $K$ is to be approximated by a sequence of degenerate kernel functions

$$
\begin{equation*}
K_{n}(t, s)=\sum_{j=1}^{n} a_{j}(t) b_{j}(s), n \geq 1 \tag{3.17}
\end{equation*}
$$

In such a way that the associated integral operators $K_{n}^{*}$ satisfy

$$
\lim _{n \rightarrow \infty}\left\|K^{*}-K_{n}^{*}\right\|=0
$$

Generally, we want this convergence to be rapid to obtain rapid convergence of $x_{n}$ to $x$ where $x_{n}$ is the solution of the approximating equation

$$
\begin{equation*}
\lambda x_{n}(t)-\int_{D} K_{n}(t, s) x_{n}(s) d s=y(t), t \in D, \quad \lambda \neq 0, \tag{3.18}
\end{equation*}
$$

There are two methods to find $x_{n}$ in Eq. (3.18).
Method 1: Substitute (3.17) into (3.18)

$$
\begin{equation*}
\lambda x_{n}(t)-\sum_{j=1}^{n} a_{j}(t) c_{j}=y(t), \tag{3.19}
\end{equation*}
$$

where $c_{j}=\int_{D} b_{j}(s) x_{n}(s) d s$ and solve for $x_{n}$ to obtain

$$
\begin{equation*}
x_{n}(t)=\frac{1}{\lambda}\left[y(t)+\sum_{j=1}^{n} a_{j}(t) c_{j}\right], t \in D, \lambda \neq 0 . \tag{3.20}
\end{equation*}
$$

To determine $c_{j}$ multiply (3.19) by $b_{i}(t)$ and integrate over $D$. This yields the system

$$
\begin{equation*}
\lambda c_{i}-\sum_{j=1}^{n} c_{j}\left\langle a_{j}, b_{j}\right\rangle=\left\langle y, b_{i}\right\rangle, i=1, \ldots, n, \tag{3.21}
\end{equation*}
$$

where $\left\langle a_{j}, b_{j}\right\rangle=\int_{a}^{b} a_{i}(t) b_{i}(t) d t$.
The system (3.21) is solved for $c_{j}$ and $x_{n}$ is obtained from (3.20).
Method 2: To find $x_{n}$ we rewrite Eq. (3.16) in the operator form

$$
\begin{equation*}
\left(\lambda-K^{*}\right) x=y, \tag{3.22}
\end{equation*}
$$

Solving Eq. (3.22), w.r.t. $x$ gives

$$
\begin{equation*}
x=\frac{1}{\lambda}\left(y+K^{*} x\right), \tag{3.23}
\end{equation*}
$$

Let $\left\{u_{i}\right\}_{i=1}^{n}$ be basis functions and

$$
K * x=c_{1} u_{1}+c_{2} u_{2}+\ldots+c_{n} u_{n}=\sum_{i=1}^{n} c_{i} u_{i},
$$

then Eq. (3.23) can be written as

$$
\begin{equation*}
x=\frac{1}{\lambda}\left(y+c_{1} u_{1}+c_{2} u_{2}+\ldots+c_{n} u_{n}\right), \tag{3.24}
\end{equation*}
$$

Substitute (3.24) into (3.22) to get

$$
\lambda\left[\frac{1}{\lambda}\left(y+c_{1} u_{1}+c_{2} u_{2}+\ldots+c_{n} u_{n}\right)\right]-K^{*}\left[\frac{1}{\lambda}\left(y+c_{1} u_{1}+c_{2} u_{2}+\ldots+c_{n} u_{n}\right)\right]=y,
$$

since operator $K^{*}$ is a linear we have

$$
\begin{equation*}
\lambda \sum_{i=1}^{n} c_{i} u_{i}-\sum_{j=1}^{n} c_{j} K * u_{j}=K * y \tag{3.25}
\end{equation*}
$$

Again by expending range of $K^{*}$ with respect to basis $\left\{u_{i}\right\}_{i=1}^{n}$

$$
\begin{equation*}
K * y=\sum_{i=1}^{n} \gamma_{i} u_{i}, K * u_{j}=\sum_{i=1}^{n} a_{i j} u_{i}, \quad j=1,2, \ldots, n \tag{3.26}
\end{equation*}
$$

and substituting (3.26) into (3.25) yields

$$
\sum_{i=1}^{n}\left\{\lambda c_{i} u_{i}-\sum_{j=1}^{n} a_{i j} c_{j}\right\}=\sum_{i=1}^{n} \gamma_{i} u_{i} .
$$

By the independence of the basis $\left\{u_{i}\right\}_{i=1}^{n}$ we obtain the linear system

$$
\begin{equation*}
\lambda c_{i} u_{i}-\sum_{j=1}^{n} a_{i j} c_{j}=\gamma_{i}, 1 \leq i \leq n . \tag{3.27}
\end{equation*}
$$

Solving Eq. (3.27) for $c_{j}$ and substituting it (3.24) we get approximate solution.

Theorem 3.5 (Atkinson [1]): Assume $\lambda-K_{n}^{*}: X \xrightarrow[\text { onto }]{1-1} X$, with $\lambda \neq 0$ and with $X=C(D)$ or $X=L_{2}(D)$ and let $K_{n}^{*}$ have a degenerate kernel (3.17). Then the linear system (3.27) is nonsingular.

Claim: The linear system (3.27) are completely equivalent with (3.22) in their solvability.

To prove statement suppose $\left(c_{1}, c_{2}, \ldots, c_{n}\right)$ is the solution of (3.27) and define $x \in X$ by using (3.24). Let us check whether $x$ satisfies the integral equation (3.22)

$$
\begin{align*}
\left(\lambda-K^{*}\right) x & =\lambda\left[\frac{1}{\lambda}\left(y+\sum_{i=1}^{n} c_{i} u_{i}\right)\right]-K *\left[\frac{1}{\lambda}\left(y+\sum_{j=1}^{n} c_{j} u_{j}\right)\right] \\
& =y+\frac{1}{\lambda}\left\{\lambda \sum_{i=1}^{n} c_{i} u_{i}-K^{*} y-\sum_{j=1}^{n} c_{j} K u_{j}\right\} \\
& =y+\frac{1}{\lambda}\left\{\sum_{i=1}^{n} \lambda c_{i} u_{i}-\sum_{i=1}^{n} \gamma_{i} u_{i}-\sum_{j=1}^{n} c_{j} \sum_{i=1}^{n} a_{i j} u_{i}\right\}  \tag{3.28}\\
& =y+\frac{1}{\lambda} \sum_{i=1}^{n}\left\{\lambda c_{i}-\gamma_{i}-\sum_{j=1}^{n} a_{i j} c_{j}\right\} u_{i}=y
\end{align*}
$$

Also, distinct coordinate vectors $\left(c_{1}, c_{2}, \ldots, c_{n}\right)$ lead distinct solution vectors $x$ in (3.22), because of the linear independence of the basis vectors $\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$. This completes the proof of the claim given above.

Now we give additional results on the solvability of the equation

$$
\begin{equation*}
\left(\lambda-K^{*}\right) x=y, \quad \lambda \neq 0, \tag{3.29}
\end{equation*}
$$

Definition 3.6: Let $K^{*}: X \rightarrow Y$. If there is a scalar $\lambda$ and an associated vector $x \neq 0$ for which $\lambda x=K^{* y}$, then $\lambda$ is called an eigenvalue and $x$ an associated eigenvector of the operator $K^{*}$. When dealing with compact operator $K^{*}$ we are interested in only the nonzero eigenvalue of $K^{*}$.

Theorem 3.7 (Fredholm alternative) (Atkinson [1]): Let $X$ be a Banach space, and let $K^{*}: X \rightarrow X$ be compact. Then the equation $\left(\lambda-K^{*}\right) x=y, \lambda \neq 0$ has a unique solution $x \in X$ iff the homogeneous equation $\left(\lambda-K^{*}\right) x=0$ has only the trivial solution $x=0$. In such a case, the operator $\lambda-K^{*}: X \xrightarrow[\text { onto }]{1-1} X$ has a bounded inverse $\left(\lambda-K^{*}\right)^{-1}$.

Theorem 3.8 (Atkinson [1]): Assume $\lambda-K^{*}: X \xrightarrow[\text { onto }]{1-1} X$ with $X$ a Banach space and $K^{*}$ bounded. Further, assume $\left\{K_{n}\right\}$ is a sequence of bounded linear operators with

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|K^{*}-K_{n}^{*}\right\|=0 . \tag{3.30}
\end{equation*}
$$

Then the operator $\left(\lambda-K^{*}\right)^{-1}$ exist from $X \rightarrow X$ for all sufficiently large $n$, say $n \geq N$ and

$$
\begin{equation*}
\left\|\left(\lambda-K_{n}\right)^{-1}\right\| \leq \frac{\left\|(\lambda-K)^{-1}\right\|}{1-\left\|(\lambda-K)^{-1}\right\|\left\|K^{*}-K_{n}^{*}\right\|}, n \geq N, \tag{3.31}
\end{equation*}
$$

For the equation $\left(\lambda-K^{*}\right) x=y$ and $\left(\lambda-K_{n}^{*}\right) x_{n}=y, n \geq N$ we have

$$
\begin{equation*}
\left\|x-x_{n}\right\| \leq\left\|(\lambda-K)^{-1}\right\|\left\|K * x-K_{n}^{*} x\right\|, n \geq N \tag{3.32}
\end{equation*}
$$

Proof: Use the identity

$$
\begin{align*}
\lambda-K_{n}^{*} & =\lambda-K^{*}+\left(K-K_{n}^{*}\right) \\
& =\left(\lambda-K^{*}\right)\left[I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-K_{n}^{*}\right)\right], \tag{3.33}
\end{align*}
$$

Choose $N$ so that

$$
\begin{equation*}
\left\|K^{*}-K_{n}^{*}\right\|<\frac{1}{\left\|(\lambda-K)^{-1}\right\|}, n \geq N \tag{3.34}
\end{equation*}
$$

Theorem 3.9 (Geometric series theorem): Let $X$ be a Banach space, and let $A$ be a bounded operator from $X$ into $X$ with

$$
\|A\|<1,
$$

Then $I-A: X \xrightarrow[\text { onto }]{1-1} X,(I-A)^{-1}$ is a bounded linear operator and

$$
\left\|(I-A)^{-1}\right\| \leq \frac{1}{1-\|A\|} .
$$

The series $(I-A)^{-1}=\sum_{j=0}^{n} A^{j}$ is called the Neumann series, under the assumption $\|A\|<1$, it converges in the space of bounded operator on $X$ to $X$. This theorem is also called the contractive mapping theorem.

Let $A=\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-K_{n}^{*}\right)$ then by using (3.34) it can be shown that

$$
\|A\|<1
$$

By the geometric series theorem the quantity $I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-K_{n}^{*}\right)$ has a bounded inverse, with

$$
\begin{equation*}
\left\|I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-K_{n}^{*}\right)\right\| \leq \frac{1}{1-\left\|\left(\lambda-K^{*}\right)^{-1}\right\|\left\|\left(K^{*}-K_{n}^{*}\right)\right\|} \tag{3.35}
\end{equation*}
$$

Using (3.33) and taking into account (3.35) we have

$$
\begin{align*}
\left\|\left(\lambda-K_{n}^{*}\right)^{-1}\right\| & =\left\|\left(\lambda-K^{*}\right)^{-1}\right\|\left\|\left[I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-K_{n}^{*}\right)\right]^{-1}\right\| \\
& \leq \frac{\left\|\left(\lambda-K^{*}\right)^{-1}\right\|}{1-\left\|\left(\lambda-K^{*}\right)^{-1}\right\|\left\|\left(K^{*}-K_{n}^{*}\right)\right\|} \tag{3.36}
\end{align*}
$$

It yields the existence of $\left(\lambda-K_{n}^{*}\right)^{-1}$ and it is bounded. For the error bound (3.32), we use the identity

$$
\begin{aligned}
x-x_{n} & =\left(\lambda-K^{*}\right)^{-1} y-\left(\lambda-K_{n}^{*}\right)^{-1} y \\
& =\left(\lambda-K_{n}^{*}\right)^{-1}\left[\left(\lambda-K_{n}^{*}\right)\left(\lambda-K^{*}\right)^{-1}-I\right] y \\
& =\left(\lambda-K_{n}^{*}\right)^{-1}\left[\left(\lambda-K_{n}^{*}\right)-\left(\lambda-K^{*}\right)\right]\left(\lambda-K^{*}\right)^{-1} y \\
& =\left(\lambda-K_{n}^{*}\right)^{-1}\left[K^{*}-K_{n}^{*}\right] x \\
& =\left(\lambda-K_{n}^{*}\right)^{-1}\left[K^{*} x-K_{n}^{*} x\right] .
\end{aligned}
$$

The error bound follows immediately.

$$
\left\|x-x_{n}\right\| \leq\left\|(\lambda-K)^{-1}\right\|\left\|K * x-K_{n}^{*} x\right\|, n \geq N
$$

A modification of the above also yields

$$
\left\|\left(\lambda-K_{n}^{*}\right)^{-1}-\left(\lambda-K^{*}\right)^{-1}\right\| \leq\left\|\left(\lambda-K_{n}^{*}\right)^{-1}\right\|\left\|\left(\lambda-K^{*}\right)^{-1}\right\|\left\|K^{*}-K_{n}^{*}\right\|
$$

From (3.30) and (3.31) this shows $\left(\lambda-K_{n}^{*}\right)^{-1} \rightarrow\left(\lambda-K^{*}\right)^{-1}$ in $L(X, X)$. This completes the proof.

1. Let $X=C(D)$ and we choose the degenerate kernel (3.17) so that the functions $a_{i}(t)$ are all continuous and the functions $b_{i}(s)$ are all absolutely integrable. To apply the above convergence theorem, note that

$$
\begin{equation*}
\left\|K-K_{n}^{*}\right\|=\max _{t \in D} \int_{D} \mid K(t, s)-K_{n}(t, s) d s, \tag{3.37}
\end{equation*}
$$

2. For $X=L_{2}(D)$, we require that all $a_{i}, b_{i} \in L^{2}(D)$. To apply the convergence theorem, we can use

$$
\begin{equation*}
\left\|K-K_{n}^{*}\right\|=\left[\iint_{D D}\left|K(t, s)-K_{n}(t, s)\right|^{2} d s d t\right]^{1 / 2} \tag{3.38}
\end{equation*}
$$

The kernels $K_{n}(t, s)$ should be chosen that $\left\|K-K_{n}^{*}\right\|$ converge to zero as rapidly as practicable.

Example 1: Let $X=Y=C[a, b]$ with $\left\|\|_{\infty}\right.$. Consider the kernel function

$$
\begin{equation*}
K(t, s)=\frac{1}{|t-s|^{\gamma}}, \tag{3.39}
\end{equation*}
$$

for some $0<\gamma<1$. Define a sequence of discontinuous kernel functions to approximate it

$$
K_{n}(t, s)= \begin{cases}\frac{1}{|t-s|^{\gamma}}, & |t-s| \geq \frac{1}{n}  \tag{3.40}\\ n^{\gamma}, & |t-s| \leq \frac{1}{n}\end{cases}
$$

For the associated integral operators,

$$
\begin{align*}
\left\|K^{*}-K_{n}^{*}\right\| & =\max _{t \in[a, b]} \int_{a}^{b}\left|K(t, s)-K_{n}(t, s)\right| d s \\
& =\max _{t \in[a, b]}\left[\left(\int_{a}^{t-1 / n}+\int_{t+1 / n}^{b}+\int_{t-1 / n}^{t+1 / n}\right)\left|K(t, s)-K_{n}(t, s)\right| d s\right],  \tag{3.41}\\
& =\max _{t \in[a, b]}\left[\int_{t-1 / n}^{t+1 / n}\left|\frac{1}{|t-s|^{\gamma}}-n^{\gamma}\right| d s\right]=\frac{2 \gamma}{1-\gamma} \frac{1}{n^{1-\gamma}} .
\end{align*}
$$

which converges to zero as $n \rightarrow \infty$.
Lemma 3.10: Let $X$ and $Y$ be normed space with $Y$ complete. Let $K \in L[X, Y]$ and $\left\{K_{n}\right\}$ be a sequence of compact operators in $L[X, Y]$ and assume $K_{n} \rightarrow K$ or $\left\|K^{*}-K_{n}^{*}\right\| \rightarrow 0$. Then $K^{*}$ is compact.

Due to Lemma 3.10, operator $K^{*}$ is compact on $C[a, b]$.

### 2.3 Taylor Series Approximation

Consider the one-dimensional IEs

$$
\begin{equation*}
\lambda x(t)-\int_{a}^{b} K(t, s) x(s) d s=y(t), t \in[a, b] \quad \lambda \neq 0 . \tag{3.42}
\end{equation*}
$$

Often we can write kernel $K(t, s)$ as a power series in $s$

$$
\begin{equation*}
K(t, s)=\sum_{i=0}^{\infty} k_{i}(t)(s-a)^{i} \tag{3.43}
\end{equation*}
$$

or in $t$

$$
\begin{equation*}
K(t, s)=\sum_{i=0}^{\infty} k_{i}(s)(t-a)^{i} . \tag{3.44}
\end{equation*}
$$

Let $K(t, s)$ denote the partial sum of the first n terms on the right side of (3.43),

$$
\begin{equation*}
K(t, s)=\sum_{i=0}^{n-1} k_{i}(t)(s-a)^{i}, \tag{3.45}
\end{equation*}
$$

Substituting (3.45) into (3.42) yields

$$
\begin{equation*}
\lambda x_{n}(t)-\sum_{i=0}^{n-1} k_{i}(t) c_{i}=y(t), \tag{3.46}
\end{equation*}
$$

where $c_{i}=\int_{a}^{b}(s-a)^{i} k_{i}(s) d s$. Solving Eq. (3.46) w.r.t $x_{n}$ gives

$$
\begin{equation*}
x_{n}(t)=\frac{1}{\lambda}\left[y(t)+\sum_{i=0}^{n-1} k_{i}(t) c_{i}\right], \tag{3.47}
\end{equation*}
$$

To find unknown $c_{i}$ we multiply Eq. (3.46) by $(t-a)^{j}$ and integrate it

$$
\begin{equation*}
\lambda c_{j}-\sum_{i=0}^{n-1} a_{i j} c_{i}=y_{j}, \quad j=0,1, \ldots, n-1, \tag{3.48}
\end{equation*}
$$

where $a_{i j}=\int_{a}^{b} k_{i}(t)(t-a)^{j} d t$.
Solving (3.48) for $c_{i}$ and substitute it into (3.47) yields approximate solution of (3.42).
Example 2: Consider the IEs

$$
\begin{equation*}
\lambda x(t)-\int_{0}^{b} e^{s t} x(s) d s=y(t), t \in[0, b], \lambda \neq 0 \tag{3.49}
\end{equation*}
$$

Solution: Write

$$
e^{s t}=\sum_{i=0}^{\infty} \frac{s^{i} t^{i}}{i!}=\sum_{i=0}^{\infty} \frac{s^{i}}{i!} t^{i}=\sum_{i=0}^{\infty} k_{i}(s) t^{i}, \quad k_{i}(s)=\frac{s^{i}}{i!} .
$$

Truncate it and substitute into (3.48) to yield

$$
\begin{equation*}
\lambda c_{j}-\sum_{i=0}^{n-1} c_{i} \frac{b^{i+j+1}}{i+j+1}=\int_{0}^{b} y(t) t^{j}, t \in[0, b], \lambda \neq 0 . \tag{3.50}
\end{equation*}
$$

And solution $x_{n}$ of the degenerate kernel equation $\left(\lambda-K_{n}^{*}\right) x_{n}=y$ is given by

$$
x_{n}(t)=\frac{1}{\lambda}\left[y(t)+\sum_{i=0}^{n-1} c_{i} \frac{t^{i}}{i!}\right],
$$

For the error analysis, let $X=C[0, b]$ with $\|\cdot\|_{\infty}$. Then

$$
\begin{align*}
\left\|K^{*}-K_{n}^{*}\right\| & =\max _{t \in[0, b]} \int_{0}^{b}\left|e^{s t}-\sum_{i=0}^{n-1} \frac{s^{i} t^{i}}{i!}\right| d s \\
& =\max _{t \in[0, b]}^{b} \int_{0}^{b} \frac{(s t)^{n}}{n!} e^{\xi_{(t, s)}} d s \leq \frac{b^{2 n+1}}{(n+1)!} e^{b^{2}} \tag{3.51}
\end{align*}
$$

This converges to zero as $n \rightarrow \infty$. By Theorem 2, we obtain convergence of $x_{n} \rightarrow x$ along with the error (3.32) whenever $(\lambda-K)^{-1}$ exists, i.e.

$$
\left\|x-x_{n}\right\| \leq \frac{b^{2 n+1}}{(n+1)!} e^{b^{2}}\left\|(\lambda-K)^{-1}\right\|\|x\|, n \geq N
$$

### 2.4 Interpolatory degenerate kernel approximation

Interpolation is a simple way to obtain kernel approximations. There are many kinds of interpolation but we consider interpolation of the kernel $K(t, s)$.

Let $\phi_{1}(t), \phi_{2}(t), \ldots, \phi_{n}(t)$ be basis for the space of interpolation functions. For example, with polynomial interpolation functions of degree $<n$, we would use

$$
\begin{equation*}
\phi_{i}(t)=t^{i-1}, 1 \leq i \leq n . \tag{3.52}
\end{equation*}
$$

Let $t_{1}, t_{2}, \ldots, t_{n}$ be interpolation nodes in the integration region $D$. The interpolation problem is as follows: Given data $y_{1}, y_{2}, \ldots, y_{n}$, find

$$
\begin{equation*}
z(t)=\sum_{j=1}^{n} c_{j} \phi_{j}(t), \tag{3.53}
\end{equation*}
$$

with

$$
\begin{equation*}
z\left(t_{i}\right)=y_{i}, 1, \ldots, n, \tag{3.54}
\end{equation*}
$$

Thus, we want to find the coefficients $c_{1}, c_{2}, \ldots, c_{n}$ solving the linear sysem

$$
\begin{equation*}
\sum_{j=1}^{n} c_{j} \phi_{j}\left(t_{i}\right)=y_{i}, i=1, \ldots, n \tag{3.55}
\end{equation*}
$$

In order for the interpolation problem to have a unique solution for all possible data $y_{1}, y_{2}, \ldots, y_{n}$, it is necessary and sufficient that

$$
\begin{equation*}
\operatorname{det}\left(\Gamma_{n}\right) \neq 0, \Gamma_{n}=\left[\phi_{j}\left(t_{i}\right)\right], \tag{3.56}
\end{equation*}
$$

With polynomial interpolation and the basis of (3.52)

$$
\begin{equation*}
\Gamma_{n}=\left[t_{i}^{j-1}\right]_{i, j=1}^{n}, \tag{3.57}
\end{equation*}
$$

This is called a Vandermonde matrix, and it is known that $\operatorname{det}\left(\Gamma_{n}\right) \neq 0$ for all distinct choices of $t_{1}, t_{2}, \ldots, t_{n}$.

To give an explicit formula for $K_{n}(t, s)$ we introduce a special basis for the interpolation method. Define $l_{k}(t)$ to be the interpolation function for which

$$
l_{k}\left(t_{i}\right)=\delta_{k i}, i=1, \ldots, n
$$

Then the solution to the interpolation problem is given by

$$
\begin{equation*}
z(t)=\sum_{j=1}^{n} l_{j}(t) y_{j}, l_{k}(t)=\prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{t-t_{i}}{t_{k}-t_{i}}, \tag{3.58}
\end{equation*}
$$

For the polynomial interpolation this is called Lagrange's form of interpolation polynomial.

Let us define

$$
\begin{equation*}
K_{n}(t, s)=\sum_{j=1}^{n} l_{j}(t) K_{n}\left(t_{j}, s\right)=\sum_{j=1}^{n} a_{j}(t) b_{j}(s) \tag{3.59}
\end{equation*}
$$

Then $K_{n}\left(t_{i}, s\right)=K\left(t_{i}, s\right), i=1,2, \ldots, n$ all $s \in D$.
Substitute (3.59) into (3.18) and using the notations $c_{j}=\int_{D} K\left(t_{j}, s\right) x_{n}(s) d s$ we have

$$
\begin{equation*}
x_{n}(t)=\frac{1}{\lambda}\left[y(t)+\sum_{j=1}^{n} l_{j}(t) c_{j}\right] \tag{3.60}
\end{equation*}
$$

To find $c_{j}$ we multiply Eq. (3.60) by $K\left(t_{j}, s\right)$ and integrate over $D$ to yield

$$
\begin{equation*}
\lambda c_{i}-\sum_{j=1}^{n} a_{i j} c_{j}=y_{i}, i=1,2, \ldots, n \tag{3.61}
\end{equation*}
$$

where $a_{i j}=\int_{D} l_{j}(s) K\left(t_{i}, s\right) d s$ and $y_{i}=\int_{D} K\left(t_{i}, s\right) y(s) d s$.

If $K(t, s) \in C^{2}[a, b]$ w.r.t. $t$ uniformly continuous for $s \in[a, b]$, then

$$
\begin{equation*}
\left\|K^{*}-K_{n}^{*}\right\|=\max _{t \in D} \int_{D}\left|K(t, s)-K_{n}(t, s)\right| d s \leq \frac{h^{2}(b-a)}{8}\left[\int_{a}^{b} \max _{t \in[a, b]}\left|\frac{\partial^{2} K(t, s)}{\partial t^{2}}\right| d s\right] . \tag{3.62}
\end{equation*}
$$

### 2.5 Projection methods (General theory)

To solve approximately the linear IE

$$
\begin{equation*}
\lambda x(t)-\int_{a}^{b} K(t, s) x(s) d s=y(t), t \in[a, b] \quad \lambda \neq 0 \tag{3.63}
\end{equation*}
$$

Let us rewrite it in the operator form

$$
\begin{equation*}
\left(\lambda-K^{*}\right) x=y \tag{3.64}
\end{equation*}
$$

where operator $K^{*}$ is assumed to be compact on a Banach space $X$ to $X$. The most popular choices are $X=C(D)$ or $X=L^{2}(D)$. In practice, we choose a sequence of finite dimensional subspace $X_{n} \subset X, n \geq 1$, with $X_{n}$ having dimension $d_{n}$. Let $X_{n}$ have a basis $\left\{\phi_{1}, \phi_{2}, \ldots, \phi_{d}\right\}$ with $d=d_{n}$ for notational simplicity. We seek a function $x_{n} \in X_{n}$ and it can be written

$$
\begin{equation*}
x_{n}(t)=\sum_{j=1}^{d} c_{j} \phi_{j}(t), t \in D \tag{3.65}
\end{equation*}
$$

This is substituted into (3.63) and coefficients $\left\{c_{1}, c_{2}, \ldots, c_{d}\right\}$ are determined by forcing the equation to be almost exact in some sense. For later use, introduce

$$
\begin{align*}
r_{n}(t) & =\lambda x_{n}(t)-\int_{a}^{b} K(t, s) x_{n}(s) d s-y(t) \\
& =\sum_{j=1}^{d} c_{j}\left\{\lambda \phi_{j}(t)-\int_{a}^{b} K(t, s) \phi_{j}(s) d s\right\}-y(t), t \in D \tag{3.66}
\end{align*}
$$

This is called residual in the approximation of the equation when using $x=x_{n}$. Symbolically,

$$
\begin{equation*}
r_{n}(t)=\left(\lambda-K^{*}\right) x_{n}-y, \tag{3.67}
\end{equation*}
$$

The coefficients $\left\{c_{1}, c_{2}, \ldots, c_{d}\right\}$ are chosen by forcing $r_{n}(t)$ to be approximately zero in some sense, then resulting function $x_{n}$ defined by (3.65) will be good approximation of the true solution $x(t)$.

## Collocation methods

Pick distinct node points $t_{1}, t_{2}, \ldots, t_{n} \in D$ and require

$$
\begin{equation*}
r_{n}\left(t_{i}\right)=0, \quad i=1, \ldots, d_{n}, \tag{3.68}
\end{equation*}
$$

This leads to determining $\left\{c_{1}, c_{2}, \ldots, c_{d}\right\}$ as the solution of the linear system

$$
\sum_{j=1}^{d} c_{j}\left\{\lambda \phi_{j}\left(t_{i}\right)-\int_{a}^{b} K\left(t_{i}, s\right) \phi_{j}(s) d s\right\}=y\left(t_{i}\right), \quad i=1, \ldots, d_{n}
$$

or

$$
\begin{equation*}
\sum_{j=1}^{d} c_{j} a_{i j}(\lambda)=y\left(t_{i}\right), i=1, \ldots, d_{n}, \tag{3.69}
\end{equation*}
$$

where $a_{i j}(\lambda)=\lambda \phi_{j}\left(t_{i}\right)-\int_{a}^{b} K\left(t_{i}, s\right) \phi_{j}(s) d s$.
Solving Eq. (3.69) we define coefficients $c_{j}$ then substitute it into (3.65) approximate solution $x_{n}$ will be defined.

An immediate question is whether this system has a solution, and if so, whether it is unique. If so, does $x_{n}$ converge $x$. To answer the questions let us introduce a projection operator $P_{n}$ that maps $X=C(D)$ onto $X_{n}$. Given $x \in C(D)$, define $P_{n} x$ that interpolates $x$ at the nodes $\left\{t_{1}, t_{2}, \ldots, t_{n}\right\}$ in the form

$$
P_{n} x(t)=\sum_{j=1}^{d} \alpha_{j} \phi_{j}(t), t \in D,
$$

with the coefficients $\alpha_{j}$ determined by solving the linear system

$$
\begin{equation*}
\sum_{j=1}^{d} \alpha_{j} \phi_{j}(t)=x\left(t_{i}\right), \quad i=\{1, \ldots, d\} \tag{3.70}
\end{equation*}
$$

This linear system has a unique solution if

$$
\begin{equation*}
\operatorname{det}\left[\phi_{j}\left(t_{i}\right)\right] \neq 0, \tag{3.71}
\end{equation*}
$$

In the case of polynomial interpolation for function of one variable, the determinant in (3.71) is referred as Vandermonde determinant. Let us consider Lagrange basis function

$$
l_{j}(t)=\prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{t-t_{j}}{t_{i}-t_{j}},
$$

with this new basis we can write

$$
\begin{equation*}
P_{n} x(t)=\sum_{j=1}^{d} l_{j}(t) x\left(t_{j}\right), t \in D \tag{3.72}
\end{equation*}
$$

Clearly $P_{n}$ is linear and finite rank. In addition as an operator on $C(D)$ to $C(D)$

$$
\begin{equation*}
\left\|P_{n}\right\|=\max _{t \in D} \sum_{j=1}^{d}\left|l_{j}(t)\right|, \tag{3.73}
\end{equation*}
$$

We note that

$$
\begin{equation*}
P_{n} z=0 \text { iff } z\left(t_{i}\right)=0, i=1, \ldots, d_{n}, \tag{3.74}
\end{equation*}
$$

The condition (3.68) can now be written as $P_{n} r_{n}=0$ which is equivalently written

$$
\begin{equation*}
P_{n}\left(\lambda-K^{*}\right) x_{n}=P_{n} y, \quad x_{n} \in X_{n}, \tag{3.75}
\end{equation*}
$$

## Galerkin's methods

Let $X=L^{2}(D)$ i.e. Hilbert space and let $\langle\cdot, \cdot\rangle$ denote the inner product for $X$, require $r_{n}$ to satisfy

$$
\begin{equation*}
\left\langle r_{n}, \phi_{i}\right\rangle=0, i=1, \ldots, d_{n}, \tag{3.76}
\end{equation*}
$$

which leads to the linear system of linear equations

$$
\begin{equation*}
\sum_{j=1}^{d} c_{j}\left[\lambda\left\langle\phi_{j}, \phi_{i}\right\rangle-\left\langle K^{*} \phi_{j}, \phi_{i}\right\rangle\right]=\left\langle y, \phi_{i}\right\rangle, i=\{1, \ldots, d\} \tag{3.77}
\end{equation*}
$$

To find $x_{n}$ we substitute the values of $c_{j}$ into Eq. (3.65). This is Galerkin's method for obtaining an approximate solution to (3.63). Similar questions arise, does this system has a solution? If so is it unique? Does $x_{n}$ converge $x$.

We note that

$$
\begin{equation*}
P_{n} z=0 \Leftrightarrow\left\langle z, \phi_{i}\right\rangle=0, i=1, \ldots, d_{n}, \tag{3.78}
\end{equation*}
$$

with $P_{n}$ we can write Eq. (3.76) as

$$
P_{n} r_{n}=0 \Leftrightarrow\left\langle r_{n}, \phi_{i}\right\rangle=0, i=1, \ldots, d_{n},
$$

or equivalently

$$
\begin{equation*}
P_{n}\left(\lambda-K^{*}\right) x_{n}=P_{n} y, \quad x_{n} \in X_{n}, \tag{3.79}
\end{equation*}
$$

Note the similarity to Eq. (3.75).

## The General framework

Let $X$ be a Banach space and $\left\{X_{n}: n \geq 1\right\}$ be a sequence of finite dimensional subspace with dimension $d_{n}$. Let $P_{n}: X \rightarrow X_{n}$ be a bounded projection operator. This means that $P_{n}$ is a bounded linear operator with

$$
P_{n} x=x, x \in X_{n},
$$

Note that this implies $P_{n}^{2} x=P_{n}\left(P_{n} x\right)=P_{n} x \Leftrightarrow P_{n}^{2}=P_{n}$, and since $\left\|P_{n} x\right\|=\max _{t \in D}\left\|P_{n} x(t)\right\|$ we have

$$
\begin{aligned}
\left\|P_{n}^{2} x\right\| & =\max _{t \in D}\left\|P_{n}^{2} x(t)\right\|=\max _{t \in D}\left\|P_{n}\left(P_{n} x(t)\right)\right\| \\
& =\max _{t \in D}\left\|P_{n} x(t)\right\|=\left\|P_{n} x\right\| \leq\left\|P_{n} x\right\|^{2}
\end{aligned}
$$

which implies

$$
\begin{equation*}
\left\|P_{n}\right\|=\left\|P_{n}^{2}\right\| \leq\left\|P_{n}\right\|^{2} \Rightarrow\left\|P_{n}\right\| \geq 1 \tag{3.80}
\end{equation*}
$$

Let us solve Eq. (3.63) using (3.79)

$$
\begin{equation*}
P_{n}\left(\lambda-K^{*}\right) x_{n}=P_{n} y, \quad x_{n} \in X_{n} . \tag{3.81}
\end{equation*}
$$

If $x_{n}$ is the solution of (3.81) then by using $P_{n} x_{n}=x_{n}$ the equation can be written as

$$
\begin{equation*}
\left(\lambda-P_{n} K^{*}\right) x_{n}=P_{n} y, \quad x_{n} \in X_{n}, \tag{3.82}
\end{equation*}
$$

The solution of Eq. (3.82) is

$$
\begin{equation*}
x_{n}=\frac{1}{\lambda}\left[P_{n} y-P_{n} K * x_{n}\right] \in X_{n}, \tag{3.83}
\end{equation*}
$$

Thus $P_{n} x_{n}=x_{n}$ leads

$$
\left(\lambda-P_{n} K^{*}\right) x_{n}=P_{n}\left(\lambda-K^{*}\right) x_{n},
$$

And this shows that (3.82) implies (3.81).
For the error analysis, we compare (3.82) with the original equation

$$
\begin{equation*}
\left(\lambda-K^{*}\right) x=y, \tag{3.84}
\end{equation*}
$$

Since both equations are defined on the original space $X$. The theoretical analysis is based on the approximation of $\lambda-P_{n} K^{*}$ by $\lambda-K^{*}$.

$$
\begin{align*}
\lambda-P_{n} K^{*} & =\left(\lambda-K^{*}\right)+\left(K^{*}-P n K^{*}\right) \\
& =\left(\lambda-K^{*}\right)\left(I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-P_{n} K^{*}\right)\right) \tag{3.85}
\end{align*}
$$

Now we prove the following theorem.
Theorem 3.11: Assume $K^{*}: X \rightarrow X$ is bounded, with $X$ a Banach space, and assume $\lambda-K^{*}: X \xrightarrow[\text { onto }]{1-1} X$. Further assume

$$
\begin{equation*}
\left\|K-P_{n} K^{*}\right\| \xrightarrow[n \rightarrow \infty]{ } 0 \tag{3.86}
\end{equation*}
$$

Then for all sufficiently large $n \geq N$, the operator $\left(\lambda-P_{n} K^{*}\right)^{-1}$ exists as a bounded operator from $X \rightarrow X$. Moreover, it is uniformly bounded

$$
\begin{equation*}
\sup _{n \geq N}\left\|\lambda-P_{n} K^{*}\right\|<\infty, \tag{3.87}
\end{equation*}
$$

For the solution of (3.82) and (3.84)

$$
\begin{equation*}
x-x_{n}=\lambda\left(\lambda-P_{n} K *\right)^{-1}\left(x-P_{n} x\right) \tag{3.88}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{|\lambda|}{\left\|\lambda-P_{n} K^{*}\right\|}\left\|x-P_{n} x\right\| \leq\left\|x-x_{n}\right\| \leq|\lambda|\left\|\left(\lambda-P_{n} K^{*}\right)^{-1}\right\|\left\|x-P_{n} x\right\|, \tag{3.89}
\end{equation*}
$$

This leads to $\left\|x-x_{n}\right\|$ converging to zero at exactly the same speed as $\left\|x-P_{n} x\right\|$.

Proof: (a) Pick $N$ such that

$$
\begin{equation*}
\varepsilon_{N}=\sup _{n \geq N}\left\|K^{*}-P_{n} K^{*}\right\|<\frac{1}{\left\|\lambda-K^{*}\right\|} . \tag{3.90}
\end{equation*}
$$

Then the inverse $\left[I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-P_{n} K^{*}\right)\right]^{-1}$ exists and is uniformly bounded by the geometric series theorem (Theorem 3.9)

$$
\left\|I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-P_{n} K^{*}\right)\right\| \leq \frac{1}{1-\varepsilon_{N}\left\|\left(\lambda-K^{*}\right)^{-1}\right\|}
$$

Using (3.85), $\left(\lambda-P_{n} K^{*}\right)^{-1}$ exists,

$$
\begin{equation*}
\left(\lambda-P_{n} K^{*}\right)^{-1}=\left[I+\left(\lambda-K^{*}\right)^{-1}\left(K^{*}-P_{n} K^{*}\right)\right]^{-1}\left(\lambda-K^{*}\right)^{-1}, \tag{3.91}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\left\|\left(\lambda-P_{n} K^{*}\right)^{-1}\right\|=\frac{\left\|\left(\lambda-K^{*}\right)^{-1}\right\|}{1-\varepsilon_{N}\left\|\left(\lambda-K^{*}\right)^{-1}\right\|}=M . \tag{3.92}
\end{equation*}
$$

This shows Eq. (3.87) holds
(b) For the error formula (3.88) multiply $\left(\lambda-K^{*}\right) x=y$ by $P_{n}$, and then re-arrange to obtain

$$
\left(\lambda-P_{n} K^{*}\right) x=P_{n} y+\lambda\left(x-P_{n} x\right) .
$$

Subtract $\left(\lambda-P_{n} K^{*}\right) x=P_{n} y$ to get

$$
\left(\lambda-P_{n} K^{*}\right)\left(x-x_{n}\right)=\lambda\left(x-P_{n} x\right)^{-1} .
$$

or

$$
\begin{equation*}
x-x_{n}=\lambda\left(\lambda-P_{n} K^{*}\right)^{-1}\left(x-P_{n} x\right) \tag{3.93}
\end{equation*}
$$

which is identical with (3.88). Taking norms and using (3.92)

$$
\begin{equation*}
\left\|x-x_{n}\right\| \leq|\lambda| M\left\|x-P_{n} x\right\| . \tag{3.94}
\end{equation*}
$$

Thus if $P_{n} x \rightarrow x$ then $x_{n} \rightarrow x$ as $n \rightarrow \infty$.
(c) The upper bound in (3.89) follows directly from (3.88), as we have just seen. The lower bound follows by taking bounds in (3.93) to obtain

$$
\begin{equation*}
|\lambda|\left\|x-P_{n} x\right\| \leq\left\|\lambda-P_{n} K *\right\|\left\|x-x_{n}\right\| . \tag{3.95}
\end{equation*}
$$

This is equivalent to the lower bound in (3.89). to obtain a lower bound that is uniform in $n$, note that $n \geq N$

$$
\left\|\lambda-P_{n} K^{*}\right\| \leq\left\|\lambda-K^{*}\right\|+\left\|K^{*}-P_{n} K\right\| \leq\left\|\lambda-K^{*}\right\|+\varepsilon_{N} .
$$

The lower bound in (3.89) can now be placed by

$$
\frac{|\lambda|}{\left\|\lambda-K^{*}\right\|+\varepsilon_{N}}\left\|x-P_{n} x\right\| \leq\left\|x-x_{n}\right\| .
$$

Combining this and (3.94) we have

$$
\begin{equation*}
\frac{|\lambda|}{\left\|\lambda-K^{*}\right\|+\varepsilon_{N}}\left\|x-P_{n} x\right\| \leq\left\|x-x_{n}\right\| \leq|\lambda| M\left\|x-P_{n} x\right\|, \tag{3.96}
\end{equation*}
$$

This shows that $x_{n}$ converges to $x$ moreover if convergence does occur, then $\left\|x-P_{n} x\right\|$ and $\left\|x-x_{n}\right\|$ tend to zero with exactly the same speed.

To apply the above theorem, we need to know whether $\| K^{*}-P_{n} K^{*}{ }_{n \rightarrow \infty} 0$. The following lemmas address this question.

Lemma 3.12 (Atkinson [1]): Let $X, Y$ be a Banach spaces, and let $A_{n}: X \rightarrow Y, n \geq 1$ be a sequence of bounded linear operators. Assume $\left\{A_{n} x\right\}$ converges for all $x \in X$. Then the convergence is uniform on compact subsets of $X$.

Lemma 3.13: Let $X$ be a Banach space, and let $\left\{P_{n}\right\}$ be a family of bounded projections on $X$ with

$$
\begin{equation*}
P_{n} x \rightarrow x, n \rightarrow \infty, x \in X . \tag{3.97}
\end{equation*}
$$

Let $K^{*}: X \rightarrow X$ be compact. Then $\left\|K^{*}-P_{n} K\right\| \rightarrow \infty, n \rightarrow \infty$.

Proof: From the definition of operator norm,

$$
\left\|K^{*}-P_{n} K\right\|=\sup _{\|x\| \leq 1}\left\|K^{*} x-P_{n} K x\right\|=\sup _{z \in K^{*}(U)}\left\|z-P_{n} z\right\|,
$$

with $K^{*}(U)=\left\{K^{*} x:\|x\| \leq 1\right\}$. The set $K^{*}(U)$ is compact. Therefore, by the preceding Lemma 3.12 and the assumption (3.97),

$$
\sup _{z \in K^{*}(U)}\left\|z-P_{n} z\right\| \rightarrow 0, n \rightarrow \infty,
$$

This proves Lemma 3.13.
Example 1: Find the rate of convergence of the projection method for the integral equation

$$
\begin{equation*}
\lambda x(t)-\int_{a}^{b} K(t, s) x(s) d s=y(t), t \in[a, b] \quad \lambda \neq 0, \tag{3.98}
\end{equation*}
$$

in the class of functions $X=C(D), X=C^{2}(D)$.
Solution: Led $D=[a, b]$ and $n \geq 1$, and define $h=\frac{b-a}{n}, t_{j}=a+j h, j=0,1, \ldots, n$. The subspace $X_{n}$ is the set of all functions for piecewise linear on $[a, b]$ with breakpoints $\left\{t_{0}, t_{1}, \ldots, t_{n}\right\}$. Introduce the Lagrange basis functions for piecewise linear interpolation

$$
l_{i}(t)=\left\{\begin{array}{l}
1-\frac{\left|t-t_{i}\right|}{h}, t_{i-1} \leq t \leq t_{i+1}  \tag{3.99}\\
0, \text { otherwise }
\end{array}\right.
$$

with the obvious adjustment of the definition for $l_{0}(t)$ and $l_{n}(t)$. The projection operator is defined by

$$
\begin{equation*}
P_{n} x(t)=\sum_{i=0}^{n} l_{i}(t) x\left(t_{i}\right) . \tag{3.100}
\end{equation*}
$$

Substitute it into (3.98) to get

$$
\lambda P_{n} x(t)-\int_{a}^{b} K(t, s) P_{n} x(s) d s=y(t), t \in[a, b] \quad \lambda \neq 0,
$$

or

$$
\begin{equation*}
\sum_{i=0}^{n}\left[l_{i}\left(t_{j}\right)-b_{i j}\right] x_{n}\left(t_{i}\right)=y\left(t_{j}\right), \quad j=0,1, \ldots, n, \tag{3.101}
\end{equation*}
$$

where $b_{i j}=\int_{a}^{b} K\left(t_{j}, s\right) l_{i}(s) d s$. Solving Eq. (3.101) for $x_{n}\left(t_{i}\right)$ and substitute it into (3.100) to get approximate solution. For convergence of $P_{n} x$ we have

$$
\left\|x-P_{n} x\right\|= \begin{cases}w(x, h), & x \in C[a, b]  \tag{3.102}\\ \frac{h^{2}}{8}\|x\|_{\infty}, & x \in C^{2}[a, b]\end{cases}
$$

This shows that $P_{n} x \rightarrow x$ for all $x \in C[a, b]$. For any compact operator $K^{*}: C[a, b] \rightarrow C[a, b]$, Lemma 3.13 implies $\left\|K^{*}-P_{n} K^{*}\right\| \rightarrow 0, n \rightarrow \infty$. For sufficiently large $n$ say $n \geq N$ the equation $\left(\lambda-P_{n} K^{*}\right) x_{n}=P_{n} y$ has a unique solution $x_{n}$ for each $y \in C[a, b]$ and

$$
\left\|x-x_{n}\right\|=\left\|\lambda ( \lambda - P _ { n } K ^ { * } ) ^ { - 1 } ( x - P _ { n } x ) \left|\leq|\lambda| M\left\|x-P_{n} x\right\| \leq|\lambda| M \omega(x, h)\right.\right.
$$

For $x \in C^{2}[a, b]$,

$$
\left\|x-x_{n}\right\|=\left\|\lambda\left(\lambda-P_{n} K^{*}\right)^{-1}\left(x-P_{n} x\right)\right\| \leq|\lambda| M \frac{h^{2}}{8}\left\|x^{\prime}\right\|_{\infty}
$$

Thus, rate of convergence of approximate method is $\left\|x-x_{n}\right\|=O\left(h^{2}\right)$.

